Bounds for the Variation of Matrix Eigenvalues and Polynomial Roots

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ABSTRACT

For two given complex matrices A, B, upper bounds are derived for the optimal matching distance between the spectra $\sigma(A)$ and $\sigma(B)$ in terms of $\|A - B\|_2$, where $\|\cdot\|_2$ is the spectral norm. The case of arbitrary matrix norms is treated. A similar result estimates the optimal matching distance between the roots of two polynomials. These bounds replace a factor of 4 in earlier results by the value $16/(3\sqrt{3}) \approx 3.08$.

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1. INTRODUCTION

Let A, B be any two complex $n \times n$ matrices. A measure for the distance between the spectra $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$, $\sigma(B) = \{\mu_1, \ldots, \mu_n\}$ is the *optimal matching distance*

$$d(\sigma(A), \sigma(B)) = \min_{\pi \in S_n} \max_{i \in \{1,\ldots,n\}} |\lambda_i - \mu_{\pi(i)}|, \tag{1}$$

where S_n denotes the group of all permutations of the numbers $1, \ldots, n$. Note that $d(\cdot, \cdot)$ is a metric in the space $\mathbb{C}^n/S_n$ of unordered n-tuples of complex numbers. The optimal matching distance will also be used to measure the distance between the roots of two polynomials f, g of degree n with complex coefficients.

We will derive upper bounds for the optimal matching distance between the spectra of A and B in terms of $\|A - B\|$. Here we distinguish between using an

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arbitrary matrix norm $\|\cdot\|$ and the important special case of the spectral norm (or operator norm) $\|\cdot\|_2$.

The prototype of such an upper bound is

$$d(\sigma(A), \sigma(B)) \leq c(n)\,(2M)^{1-1/n}\,\|A - B\|^{1/n}, \tag{2}$$

where $M = \max\{\|A\|, \|B\|\}$ and $c(n)$ is a constant growing with the dimension n. The best results of this type were given by Bhatia, Elsner, and Krause [3], who showed that $c(n) \leq 4 \cdot 2^{-1/n} \cdot n^{1/n}$, where the factor $n^{1/n}$ can be dropped in the case of the spectral norm $\|\cdot\|_2$.

Bounds for the optimal matching distance between the roots of two polynomials f, g in terms of the magnitude of the coefficients of $f - g$ are derived in a similar manner. An early result of this type was due to Ostrowski [8, Appendix A]. He showed that for the polynomials

$$f(x) = x^n + \sum_{i=1}^{n} \alpha_i\, x^{n-i} = \prod_{i=1}^{n} (x - \lambda_i), \tag{3}$$

$$g(x) = x^n + \sum_{i=1}^{n} \beta_i\, x^{n-i} = \prod_{i=1}^{n} (x - \mu_i) \tag{4}$$

the optimal matching distance is bounded as

$$d(\{\lambda_1, \ldots, \lambda_n\}, \{\mu_1, \ldots, \mu_n\}) \leq (2n-1)\Big(\sum_{i=1}^{n} |\alpha_i - \beta_i|\, \gamma^{n-i}\Big)^{1/n}, \tag{5}$$

where $\gamma = 2\max_i\{|\alpha_i|^{1/i}, |\beta_i|^{1/i}\}$. A significant improvement was given by Bhatia, Elsner, and Krause [3], who showed that the factor $2n - 1$ can be replaced by the constant 4, independent of the degree n.

We will show in this paper that the factor 4 in the mentioned results can be replaced by the value $16/(3\sqrt{3}) \approx 3.08$.

2. PRELIMINARY RESULTS

It is a very useful tool to have a max-min characterization of the optimal matching distance apart from the min-max formula (1). Let $|S|$ denote the number of elements of a finite set S.

LEMMA 1. *Let $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$ and $\boldsymbol{\mu} = \{\mu_1, \ldots, \mu_n\}$ be two unordered n-tuples of complex numbers. Define $d(\boldsymbol{\lambda}, \boldsymbol{\mu})$ as in (1) and*

$$\tilde{d}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \max_{\substack{I,J \subseteq \{1,\ldots,n\} \\ |I|+|J|=n+1}} \min_{\substack{i \in I \\ j \in J}} |\lambda_i - \mu_j|. \tag{6}$$

Then $d(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \tilde{d}(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Proof. Define the $n \times n$ matrix $D = (d_{ij})$ by $d_{ij} = |\lambda_i - \mu_j|$. For every permutation $\pi \in S_n$ the set $\{d_{1,\pi(1)}, \ldots, d_{n,\pi(n)}\}$ gives a "diagonal" of D. We use the Frobenius-König theorem (see e.g., [7, p. 97]):

For any $n \times n$ matrix D and $x \geq 0$ the following conditions are equivalent:

(i) *Every diagonal of D contains at least one element greater than or equal to x.*

(ii) *D has an $r \times s$ submatrix containing only elements greater than or equal to x with $r + s = n + 1$.*

By definition $d(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is the maximal value of x such that condition (i) is satisfied, and $\tilde{d}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is the maximal value of x such that condition (ii) is satisfied. The equivalence of the two conditions yields $d(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \tilde{d}(\boldsymbol{\lambda}, \boldsymbol{\mu})$. ∎

The following two lemmas are the key to obtaining improvements over the results in [3].

LEMMA 2. *Let $\lambda_1, \ldots, \lambda_n$ and μ be $n+1$ points in the complex plane, and let ξ be a continous curve joining λ_1 and μ. If $|\lambda_i - \mu| \geq \Delta > 0$ for $i = 1, \ldots, r \leq n$, then there exists a point $x \in \xi$ such that*

$$\prod_{i=1}^{n} |x - \lambda_i| \geq a_{n,r}\, \Delta^n, \tag{7}$$

where

$$a_{n,r} = \min_{\beta_1,\ldots,\beta_{n-r}} \max_{t\in[0,1]} \left| t^n - \sum_{j=1}^{n-r} \beta_j t^{n-j} \right| \tag{8}$$

$$\geq \left[\sqrt{2n+1} \binom{2n}{n-r} \right]^{-1}. \tag{9}$$

Proof. Consider the map $m : \mathbb{C} \to [0, \Delta]$ defined by

$$m(x) = \max\{\Delta - |x - \mu|,\ 0\}.$$

Obviously m is a contraction, i.e., for $x, y \in \mathbb{C}$

$$|m(x) - m(y)| \leq |x - y|.$$

Note that $m(\lambda_1) = m(\lambda_2) = \cdots = m(\lambda_r) = 0$ and $m(\mu) = \Delta$. Thus the image $m(\xi)$ of the continuous curve ξ is the entire interval $[0, \Delta]$. We have

$$\begin{aligned}\max_{x\in\xi}\prod_{i=1}^{n}|x-\lambda_i| &\geq \max_{t\in[0,\Delta]}\prod_{i=1}^{n}|t-m(\lambda_i)| \\ &= \max_{t\in[0,\Delta]} t^r \prod_{i=r+1}^{n}|t-m(\lambda_i)| \\ &\geq \Delta^n \min_{\alpha_1,\ldots,\alpha_{n-r}\in[0,1]}\ \max_{t\in[0,1]} t^r\prod_{i=1}^{n-r}|t-\alpha_i|.\end{aligned}$$

This proves (8), because the min-max term can also be written as

$$a_{n,r} = \min_{\beta_1,\ldots,\beta_{n-r}}\ \max_{t\in[0,1]}\left|t^n - \sum_{j=1}^{n-r}\beta_j t^{n-j}\right|,$$

which is a Chebyshev approximation problem on the interval [0,1]. The case $r = 0$ is well known, and the explicit solution leads to Chebyshev polynomials: $a_{n,0} = 2^{1-2n}$. But if $r > 0$, then a general solution can be calculated only in the L_2-norm by means of Hilbert space theory (see e.g., Achieser [1, Chapter I, §15]):

$$\min_{\beta_1,\ldots,\beta_{n-r}}\left(\int_0^1\left(t^n-\sum_{j=1}^{n-r}\beta_j t^{n-j}\right)^2 dt\right)^{1/2} = \left[\sqrt{2n+1}\binom{2n}{n-r}\right]^{-1}.$$

This is a lower bound on $a_{n,r}$, because the sup norm on [0, 1] always majorizes the L_2-norm. ■

In the following lemma we will use the notation of upper $\lceil\cdot\rceil$ and lower $\lfloor\cdot\rfloor$ Gauss brackets.

LEMMA 3. *For $i = 1,\ldots,n$ let ξ_i be continuous curves in the complex plane with end points λ_i and μ_i. Let $\Delta = d(\{\lambda_1,\ldots,\lambda_n\},\{\mu_1,\ldots,\mu_n\})$. Then there exists a point x on one of these curves such that*

$$\Delta \leq b_n\left(\max\left\{\prod_{i=1}^{n}|x-\lambda_i|,\prod_{i=1}^{n}|x-\mu_i|\right\}\right)^{1/n}, \tag{10}$$

where

$$b_n = \left(a_{n,\lceil(n+1)/2\rceil}\right)^{-1/n}, \tag{11}$$

and $a_{n,k}$ is defined as in (8). Furthermore $b_n \leq c_n$, where

$$c_n = \left[\sqrt{2n+1}\binom{2n}{\lfloor\frac{n-1}{2}\rfloor}\right]^{1/n} \tag{12}$$

and c_n can be bounded by a constant independent of n:

$$c_n \le \frac{16}{3\sqrt{3}}. \tag{13}$$

Proof. From Lemma 1 we know that there are sets of indices $I, J \subseteq \{1, \ldots, n\}$ such that $|I| + |J| = n + 1$ and

$$\Delta \le |\lambda_i - \mu_j|, \quad i \in I,\ j \in J.$$

Assume without loss of generality that 1 is an index in the nonempty intersection of I and J.

Consider the case $|I| \ge |J|$: We can assume (by appropriate numbering) that $\{1, \ldots, r\} \subseteq I$ for $r = \lceil (n+1)/2 \rceil$. Now $\lambda_1, \ldots, \lambda_n$, $\mu = \mu_1$, $\xi = \xi_1$, and Δ satisfy the conditions of Lemma 2. Hence there is an $x \in \xi_1$ such that

$$\prod_{i=1}^{n} |x - \lambda_i| \ge a_{n,\lceil (n+1)/2 \rceil}\, \Delta^n.$$

If, on the other hand, $|I| \le |J|$, then we obtain similarly

$$\prod_{i=1}^{n} |x - \mu_i| \ge a_{n,\lceil (n+1)/2 \rceil}\, \Delta^n$$

for some $x \in \xi_1$. In any case there exists an $x \in \xi_1$ such that

$$\Delta \le \left(a_{n,\lceil (n+1)/2 \rceil}\right)^{-1/n} \max\left\{\prod_{i=1}^{n} |x - \lambda_i|, \prod_{i=1}^{n} |x - \mu_i|\right\}^{1/n},$$

and (10) follows.

From the second part of Lemma 2 we have the inequality

$$a_{n,\lceil (n+1)/2 \rceil} \ge \left[\sqrt{2n+1}\binom{2n}{n - \lceil \frac{n+1}{2} \rceil}\right]^{-1},$$

and thus

$$b_n = \left(a_{n,\lceil (n+1)/2 \rceil}\right)^{-1/n} \le \left[\sqrt{2n+1}\binom{2n}{\lfloor \frac{n-1}{2} \rfloor}\right]^{1/n} = c_n.$$

To estimate c_n, we consider the binomial coefficient

$$\binom{2n}{\lfloor \frac{n-1}{2} \rfloor} < \binom{2n}{\frac{n}{2}} = \frac{(2n)!}{(\frac{n}{2})!(\frac{3n}{2})!}. \tag{14}$$

In case of n being odd, the factorials of the noninteger numbers in the denominator are to be interpreted in the sense of the Γ-function. We apply the right hand inequality of the Stirling formula

$$\left(1+\frac{1}{12k}\right)\sqrt{2\pi k}\left(\frac{k}{e}\right)^k < k! < \left(1+\frac{1}{12k}+\frac{1}{288k^2}\right)\sqrt{2\pi k}\left(\frac{k}{e}\right)^k$$

to the factorial in the numerator of the fraction in (14) and the left hand inequality to the factorials in the denominator. We obtain

$$\binom{2n}{n/2} < \frac{1+\frac{1}{24n}+\frac{1}{1152n^2}}{1+\frac{2}{9n}+\frac{1}{108n^2}} \cdot \frac{2}{\sqrt{3\pi n}} \cdot \left(\frac{16}{3\sqrt{3}}\right)^n,$$

and the first fraction on the right hand side is less than 1; hence

$$\sqrt{2n+1}\binom{2n}{n/2} < \sqrt{\frac{8n+4}{3\pi n}}\left(\frac{16}{3\sqrt{3}}\right)^n.$$

One can easily verify that the inequality

$$\sqrt{2n+1}\binom{2n}{n/2} < \left(\frac{16}{2\sqrt{3}}\right)^n$$

is valid for $n = 1$ and $n = 2$. If, on the other hand, $n \geq 3$ then $(8n+4)/(3\pi n) < 1$, and this inequality follows from the one before. Taking the nth root, this proves the remaining assertion. ■

3. MAIN RESULTS

THEOREM 1. *For any two matrices $A, B \in \mathbb{C}^{n,n}$*

$$d(\sigma(A), \sigma(B)) \leq b_n (2M_2)^{1-1/n} \|A-B\|_2^{1/n}, \tag{15}$$

where $M_2 = \max\{\|A\|_2, \|B\|_2\}$, and b_n, defined in (11), *is bounded by*

$$b_n \leq \frac{16}{3\sqrt{3}}.$$

Proof. The eigenvalues of the convex combination $C(t) = (1-t)A + tB$, $t \in [0, 1]$, can be considered as continuous curves $\xi_1(t), \ldots, \xi_n(t)$ in the complex plane parametrized by $t \in [0, 1]$. These curves are generally not unique, because

$C(t)$ may have multiple eigenvalues for some $t \in [0, 1]$. However, we choose one arbitrary set of such curves. Let $\lambda_i = \xi_i(0)$ and $\mu_i = \xi_i(1)$, $i = 1, \ldots, n$, be the eigenvalues of A and B, respectively, and let $\Delta = d(\sigma(A), \sigma(B))$ be the optimal matching distance between the spectra of A and B. Apply Lemma 3 to obtain

$$\Delta \leq b_n \left(\max \left\{ \prod_{i=1}^{n} |x - \lambda_i|, \prod_{i=1}^{n} |x - \mu_i| \right\} \right)^{1/n} \tag{16}$$

for some x on one of the curves ξ_i, say $x = \xi_k(t_0)$, $t_0 \in [0, 1]$.

Elsner [4] showed that for any two matrices $X, Y \in \mathbb{C}^{n,n}$ and any eigenvalue α of Y

$$|\det(X - \alpha I)| \leq \|X - Y\|_2 \left(\|X\|_2 + \|Y\|_2\right)^{n-1}.$$

Substitute $X = A$, $Y = C(t_0) = (1 - t_0)A + t_0 B$, and $\alpha = x$ to obtain

$$\begin{aligned} \prod_{i=1}^{n} |x - \lambda_i| &= \left|\det[A - \xi_k(t_0)I]\right| \\ &\leq \|A - C(t_0)\|_2 \left(\|A\|_2 + \|C(t_0)\|_2\right)^{n-1} \\ &\leq t_0 \|A - B\|_2 \left[(2 - t_0)\|A\|_2 + t_0\|B\|_2\right]^{n-1} \\ &\leq \|A - B\|_2 \left(2 \max\{\|A\|_2, \|B\|_2\}\right)^{n-1}, \end{aligned}$$

and by a similar calculation

$$\prod_{i=1}^{n} |x - \mu_i| \leq \|A - B\|_2 \left(2 \max\{\|A\|_2, \|B\|_2\}\right)^{n-1}.$$

Combine both inequalities with (16) to complete the proof. ■

THEOREM 2. *For any two matrices $A, B \in \mathbb{C}^{n,n}$ and any matrix norm $\|\cdot\|$*

$$d(\sigma(A), \sigma(B)) \leq \tilde{b}_n \,(2M)^{1-1/n} \,\|A - B\|^{1/n}, \tag{17}$$

where $M = \max\{\|A\|, \|B\|\}$ and

$$\tilde{b}_n = b_n \cdot n^{1/n} \leq 3.46\,, \quad n \in \mathbb{N}, \qquad \lim_{n \to \infty} \tilde{b}_n \leq \frac{16}{3\sqrt{3}}.$$

Proof. Again for an eigenvalue $x = \xi_k(t_0)$ of $C(t_0) = (1-t_0)A+t_0B$, $t_0 \in [0, 1]$, the inequality (10) holds.

From Friedland [5] we know that for any two matrices $X, Y \in \mathbb{C}^{n,n}$ and any matrix norm $\|\cdot\|$

$$|\det X - \det Y| \leq n \max\{\|X\|, \|Y\|\}^{n-1} \|X - Y\|.$$

Substitute $X = A - xI$ and $Y = C(t_0) - xI$, and note that $\det Y = 0$. It is easy to verify that $\max\{\|X\|, \|Y\|\} \leq 2\max\{\|A\|, \|B\|\}$ and $\|X - Y\| \leq \|A - B\|$; hence

$$\begin{aligned}\prod_{i=1}^{n} |x - \lambda_i| &= |\det X - \det Y| \\ &\leq n \max\{\|X\|, \|Y\|\}^{n-1} \|X - Y\| \\ &\leq n \cdot (2M)^{n-1} \|A - B\|.\end{aligned}$$

Clearly the same upper bound holds for $\prod_{i=1}^{n} |x - \mu_i|$. Combine this with (10) to obtain

$$\Delta \leq b_n\, n^{1/n}\, (2M)^{1-1/n}\, \|A - B\|^{1/n}.$$

Finally note that

$$\lim_{n\to\infty} b_n\, n^{1/n} = \lim_{n\to\infty} b_n \leq \frac{16}{3\sqrt{3}}.$$

Numerical evaluation of $b_n\, n^{1/n}$ for $n \leq 29$ gives a maximum value 3.46, which is attained for $n = 9$. If, on the other hand, $n > 29$, then $b_n\, n^{1/n} \leq 16/(3\sqrt{3})\, n^{1/n} <$ 3.46. ∎

THEOREM 3. *Let f and g be two polynomials defined as in* (3) *and* (4). *Then the optimal matching distance $\Delta = d(\{\lambda_1, \ldots, \lambda_n\}, \{\mu_1, \ldots, \mu_n\})$ is bounded as*

$$\Delta \leq b_n \left(\sum_{i=1}^{n} |\alpha_i - \beta_i|\, \gamma^{n-i} \right)^{1/n}, \tag{18}$$

where

$$\gamma = 2 \max_{k\in\{1,\ldots,n\}} \max\left\{ |\alpha_k|^{1/k}, |\beta_k|^{1/k} \right\}.$$

Proof. The roots of the convex combination $(1-t)f + tg$ can again be considered as continuous curves ξ_i, $i = 1, \ldots, n$, in the complex plane with end points λ_i respectively μ_i. Again we apply Lemma 3 and find a point $x = \xi_k(t_0)$ for which (10) is true. The fact that x is a root of $(1 - t_0)f + t_0 g$ yields

$$\begin{aligned}\prod_{i=1}^{n} |x - \lambda_i| &= |f(x)| \\ &= \left| f(x) - [(1 - t_0)f(x) + t_0 g(x)] \right| \\ &= t_0\, |f(x) - g(x)| \\ &\leq \left| \sum_{i=1}^{n} (\alpha_i - \beta_i)\, x^{i-k} \right|.\end{aligned}$$

TABLE 1.

n odd			n even		
n	b_n	c_n	n	b_n	c_n
1	1.0000	1.7321	2	1.0000	1.4953
3	2.1137	2.5132	4	1.9005	2.2134
5	2.4531	2.7213	6	2.2623	2.4893
7	2.6154	2.8176	8	2.4546	2.6329
9	2.7105	2.8731	10	2.5735	2.7204
11	2.7731	2.9091	12	2.6543	2.7794
13	2.8174	2.9344	14	2.7128	2.8217
15	2.8504	2.9532	16	2.7571	2.8536
17	2.8760	2.9677	18	2.7918	2.8785
19	2.8964	2.9791	20	2.8197	2.8984

Clearly a similar calculation yields

$$\prod_{i=1}^{n} |x - \mu_i| \leq \left|\sum_{i=1}^{n} (\alpha_i - \beta_i)\, x^{i-k}\right|.$$

It is known (see Ostrowski [8]) that the modulus of any root of the convex combination $(1 - t)f + tg$ is less than or equal to γ. Apply this to x in the above inequalities and combine them with (10) to complete the proof. ∎

4. REMARKS

1. In the polynomial case there are examples [3] which show that in Theorem 3 the factor 3.08 cannot be replaced by a value smaller than 2.

2. It is reasonable to calculate values of b_n numerically by solving the approximation problem (8). In Table 1 for $n \leq 20$ values of b_n are listed and compared with c_n, defined in (12).

3. The material of this paper is based, in part, on my Ph.D. dissertation [6]. There and in the book of R. Bhatia [2] one can find surveys of results of this type and their history.

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